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Some Spectral Properties of Subnormal Operators and Related Operators

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Abstract

Subnormal operator on a Hilbert space and its spectra have been studied by many authors. The aim of this work is to present some ideas on how to obtain new results related to the some spectral properties of the subnormal operator. To show that we have used a particular type of operators as an example, which is represented by certain infinite matrix. We have reached new results that have been presented in the form of theorems.

Keywords: Subnormal operator, hyponormal operator, spectrum, infinite matrices, sequence spaces.

* Introduction

In this work, an interesting type of operators is the focus of attention, which is today called subnormal. From the literature presented in [4,5,6,7], it is noted that, several authors have studied the spectra of the subnormal operator on Hilbert space. Surprisingly, to the authors' knowledge, we did not find a study that determines the spectra for a specific example of this type. Our main problem is devoted to answer this question:

How to apply some obtained results related to the spectra of a generalized difference operator for getting other results associated with spectra of the subnormal operator?

We use some known methods to get the adjoint of operator and we consider three methods for classifying the spectrum.

The concept of subnormality has been introduced by Halmos [11] and has been defined by others, see [4],[5], [7]. In fact, as it was mentioned, the concept of subnormality is very close to concept of normality, where subnormal operator arises from the concept of a normal operator.

The spectrum of some operators on certain spaces has been studied by many authors, see [1],[2], [9], [19].

Let X be a complex infinite dimensional Banach space and $B(X)$ be the set of all bounded linear operators on X into itself. If $T \in B(X)$, we use $R(T)$ to denote the range of T.

For a Banach space X we use X^* to denote the dual space of X.
If $T \in B(X)$, then $T^* \in B(X^*)$ is the adjoint operator of T.

We begin by giving the definitions of some basic concepts, which are needed in this work.

*** Definition 1.1**

[10]. The Hahn space h is defined by $h = \left\{x = (x_k)_1^\infty : \lim_{k\to\infty} x_k = 0 \text{ and } \sum_{k=1}^\infty k|x_{k+1} - x_k| < \infty\right\}$, with the norm

$$\|x_k\|_h = \sum_{k=1}^\infty k|\Delta x_k| + \sup_k |x_k|.$$

Or with a new norm, see [16, Proposition 2.1] as $\|x_k\|_h = \sum_{k=1}^\infty k|\Delta x_k|$, where $\Delta x_k = |x_{k+1} - x_k|$.

The space σ_∞ of all absolutely summable sequences $x = (x_k)_0^\infty$ is defined as

$$\sigma_\infty = \left\{x = (x_k)_0^\infty : \sup_{n\in\mathbb{N}} \frac{1}{n+1} \left|\sum_{k=0}^n x_k\right| < \infty\right\}.$$

The spaces h and σ_∞ are Banach spaces and $h^* \cong \sigma_\infty$ [10,12].

*** Theorem 1.1**

[10,12,13]. The matrix $A = (a_{nk})$ gives rise to abounded linear operator $T \in B(h)$ if and only if

1- $\lim_{n\to\infty} a_{nk} = 0$, for all $k = 1,2, \ldots$,

2- $\sum_{n=1}^\infty n\left|a_{nk} - a_{n+1,k}\right|$ converges, for all $k = (1,2, \ldots)$,

3- $\sup_k \frac{1}{k} \sum_{n=1}^\infty n\left|\sum_{v=1}^k (a_{nv} - a_{n+1,v})\right| < \infty$.

*** Definition 1.2**

If $T \in B(X)$, with T we associate the operator
$T_\lambda = T - \lambda I$,

where I is the identity mapping of X onto itself. If T_λ has an inverse which is linear, we denote it by T_λ^{-1} and call it the resolvent operator of T. All of the points λ in the complex plane $\mathbb{C}$ are divided into two mutually exclusive and complementary sets:

The resolvent set. $\rho(T, X) = \{\lambda \in \mathbb{C}: T - \lambda I \text{ is a bijection}\}$, and

The spectrum of T : $\sigma(T, X) = \{\lambda \in \mathbb{C}: T - \lambda I \text{ is not invertible}\}$,

The spectrum $\sigma(T, X)$ is the complement of $\rho(T, X)$ in the complex plane $\mathbb{C}$, $\sigma(T, X)$ can be analyzed into three disjoint sets as follows:

The point spectrum: $\sigma_p(T,X) = \{\lambda \in \mathbb{C}: T - \lambda I \text{ is not injective}\}$,

The continuous spectrum:

$\sigma_c(T,X) = \{\lambda \in \mathbb{C}: T - \lambda I \text{ is injective and } \overline{R(T-\lambda I)} = X, \text{but } R(T-\lambda I) \neq X\}$,

The residual spectrum:

$\sigma_r(T,X) = \{\lambda \in \mathbb{C}: T - \lambda I \text{ is injective, but } \overline{R(T-\lambda I)} \neq X\}$,

These three subspectra form disjoint subdivisions

$\sigma(T,X) = \sigma_p(T,X) \cup \sigma_c(T,X) \cup \sigma_r(T,X)$.

Three more subdivisions of the spectrum which are not necessarily disjoint can be defined, see Appel et al. [3].

The approximate point spectrum of T

$\sigma_{ap}(T,X) = \{\lambda \epsilon \mathbb{C}: \text{there exists a Weyl sequence for } T - \lambda I\}$,

where the sequence (x_k) in X is called a Weyl sequence for T if $\|x_k\| = 1$ and $\|Tx_k\| \to 0$, as $\kappa \to \infty$.

The defect spectrum of T : $\sigma_\delta(T,X) = \{\lambda \in \mathbb{C}: \mathcal{R}(\lambda I - T) \neq X\}$

The compression spectrum : $\sigma_{co}(T,X) = \{\lambda \in \mathbb{C}: \overline{\mathcal{R}(\lambda I - T)} \neq X\}$,

The two subspectra $\sigma_{ap}(T,X)$ and $\sigma_\delta(T,X)$ are not necessarily disjoint. As well as $\sigma_{ap}(T,X)$ and $\sigma_{co}(T,X)$ are not necessarily disjoint.

Where$\sigma(T,X) = \sigma_{ap}(T,X) \cup \sigma_\delta(T,X)$,

$\sigma(T,X) = \sigma_{ap}(T,X) \cup \sigma_{co}(T,X)$.

Clearly, $\sigma_p(T,X) \subseteq \sigma_{ap}(T,X)$ and $\sigma_{co}(T,X) \subseteq \sigma_\delta(T,X)$.

Since$\sigma(T,X) = \sigma_p(T,X) \cup \sigma_c(T,X) \cup \sigma_r(T,X)$, so, we have$\sigma_r(T,X) = \sigma_{co}(T,X) \setminus \sigma_p(T,X)$,

$\sigma_r(T,X) = \sigma(T,X) \setminus [\sigma_p(T,X) \cup \sigma_{co}(T,X)]$.

Also, by proposition in [3] we have the following relations:

$\sigma(T^*,X^*) = \sigma(T,X)$,

$\sigma_{ap}(T^*,X^*) = \sigma_\delta(T,X)$,

$\sigma_\delta(T^*,X^*) = \sigma_{ap}(T,X)$,

$\sigma_p(T^*,X^*) = \sigma_{co}(T,X)$,

$\sigma(T,X) = \sigma_{ap}(T,X) \cup \sigma_p(T^*,X^*) = \sigma_p(T,X) \cup \sigma_{ap}(T^*,X^*)$.

There are another classification of the spectrum, a linear operator with domain and range in a normed space X is classified into I, II or III ,see Taylor and Halberg [17-18] to get details. There are some relation

$\sigma_p(T,X) = I_3(T,X) \cup II_3(T,X) \cup III_3(T,X)$,

$\sigma_r(T,X) = III_1(T,X) \cup III_2(T,X)$

$\sigma_c(T,X) = II_2(T,X)$

$\sigma_{ap}(T,X) = \sigma(T,X) \setminus III_1(T,X)$,

$\sigma_\delta(T,X) = \sigma(T,X) \setminus I_3(T,X)$

Now, we present some concepts, which are associated with a subnormal operator and its normal extension.

*** Definition 1.3.**

An operator S acting on a Hilbert space H is said to be subnormal if there is a Hilbert space K containing H and a normal operator N on K such that $NH \subseteq H$ and $S = N \restriction_H$, and N is called a normal extension of S and S is the restriction of N.

Equivalently, S is subnormal on a subspace H of K, if the normal operator N, acting on K leaves H invariant, and S is the restriction of N to H.

An example of a subnormal operator is the unilateral shift, such that, the bilateral shift is a normal extension. If U is a bilateral shift relative to the spaces $\{K_n\}$ and $H = K_0 \oplus K_1 \oplus \cdots$, then H is invariant for U and $S = U \restriction_H$ is a unilateral shift [6].

In particular, every normal operator is subnormal [6].

The more so, as the behavior of some subnormal operators is rather startlingly different from that of the normal operators.

*** Definition 1.4.**

An operator S is called quasinormal if S and S^*S commute. Every normal operator is quasinormal but the converse is obviously false [5].

Every quasinormal operator is subnormal [5], [6].

*** Theorem 1.2.**

If $S \in B(H)$, then the following statements are equivalent:

(a) S is subnormal,

(b) S has a quasinormal extension.

*** Proof**

If S is subnormal, then S has a normal extension N , it follows that N is a quasinormal extension for S , because $N(N^*N) = (NN^*)N = (N^*N)N$.

Conversely, if (b) holds, then S is subnormal. Thus (a) and (b) are equivalent.

Also, in 1950 Paul Halmos [11] introduced a larger class of operators, which are called hyponormal.

*** Definition 1.5.**

An operator A is hyponormal if $A^*A \geq AA^*$.

*** Theorem 1.3.**

Every subnormal operator is hyponormal.

*** Proof**

This theorem has been proved in [5], [6], but we can prove it with other way: If S (on H) is subnormal, with normal extension N (on K), P is projection from K onto H,if $f \in H$, then $\|S^*f\| = \|PN^*f\| \leq \|N^*f\| = \|Nf\| = \|Sf\|$, it is equivalent to the operator inequality $SS^* \leq S^*S$.

Indeed, the results in this paper, generally speaking, are concerned with the relationships that exist between the spectrum of a subnormal operator and that of its normal extension.

*** Theorem 1.4**

[8] Suppose that T is a normal operator and that λ is a complex number. Then λ is not an eigenvalue for T if and only if $(T-\lambda I)(H)$ is dense in H.

*** Theorem 1.5**

[8] Let $T : H \to H$ be a normal operator on a Hilbert space, then $\sigma_r(T) = \emptyset$.

*** Theorem 1.6**

[7] If S is a subnormal operator and N its minimal normal extension, then $\sigma(N) \subseteq \sigma(S)$, $\sigma_p(S) \subseteq \sigma_P(N)$ and $\sigma_{ap}(S) \subseteq \sigma(N)$.

*** Theorem 1.7**

[6] If A is a hyponormal operator, then $\sigma(A) = \sigma_r(A)$.

*** Theorem 1.8**

[6] If A is a hyponormal operator and λ is an isolated point of $\sigma(A)$
,then $\lambda \in \sigma_P(A)$.

*** Theorem 1.9**

[6] If A is a hyponormal operator and $\lambda \in \sigma_P(A)$, then $\ker(A-\lambda)$ reduces A.

*** Main Results**

The principal part is devoted to description of certain operator as example of a subnormal operator and determine its spectrum and fine spectrum.

We will introduce the spectra analysis of a generalized difference operator B(a) on the Hahn sequence space h.

The generalized difference operator $B(a): \mu \to \mu$ is defined on the Banach sequence space μ as:

$$B(a)x := (ax_0, ax_1, ax_2, \dots),\ x = (x_k)_{k=0}^{\infty} \in \mu,$$

where $a \in \mathbb{R},\ a \neq 0$.

This operator can be represented by a band matrix as

$$B(a) = \begin{bmatrix} a & 0 & 0 & \cdots \\ 0 & a & 0 & \cdots \\ 0 & 0 & a & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} .$$

It is clear that, this operator is normal, so it is subnormal.

Now, we will present all results of this work in the following theorems. The first theorem shows the bounded linearity of the operator B(a) on h.

*** Theorem 2.1**

The operator $B(a): h \to h$ is a bounded linear operator.

*** Proof**

The linearity of B(a) is trivial and so is omitted here for brevity. To show the operator B(a) is a bounded linear transformation on h into itself, it is enough to prove that B(a) satisfies the three conditions given by Theorem 1.1.

Obviously, the matrix $B(a) = (b_{nk})$ satisfies

$$\lim_{n\to\infty} b_{nk} = 0, k = 1,2,3, ...$$

Also, let

$$R_k = \sum_{n=1}^{\infty} n\left|b_{nk} - b_{n+1,k}\right|, \quad k = 1,2,3, ...$$

So

$R_1 = |a|,$

$R_2 = |-a| + 2|a| = 3|a|,$

and

$R_3 = 2|-a| + 3|a| = 5|a|.$

Then, in general, we obtain

$R_k = \sum_{n=1}^{\infty} n\left|b_{nk} - b_{n+1,k}\right| = (k-1)|a| + k|a|,$

which is convergent, for each fixed $k \in \mathbb{N}$.

Additionally, let

$$S_k = \sum_{n=1}^{\infty} n\left|\sum_{v=1}^{k}(b_{nv} - b_{n+1,v})\right|, \quad k = 1,2,3, \cdots.$$

So

$S_1 = |a|,$

$S_2 = 2|a|$

and

$S_3 = 3|a|$

Precisely, we have the following cases:

For $n = 1$,

$$\left|\sum_{v=1}^{k}(b_{1v} - b_{2v})\right| = |0| = 0.$$

For $n = k > 1$,

$$\left|\sum_{v=1}^{k}(b_{kv} - b_{k+1,v})\right| = |a|.$$

For $n = k + 1 > 1$,

$$\left|\sum_{v=1}^{k}(b_{k+1,v} - b_{k+2,v})\right| = |0| = 0$$

But, for $n \neq 1, k, k+1$, we have

$$\left|\sum_{v=1}^{k}(b_{nv} - b_{n+1,v})\right| = |0| = 0.$$

Then, for $k \geq 2$

$$S_k = \sum_{n=1}^{\infty} n\left|\sum_{v=1}^{k}(b_{nv} - b_{n+1,v})\right| = |0| + k|a| + (k+1)|0| = k|a|.$$

Thus

$$\sup_{k\geq 2} \frac{1}{k}\sum_{n=1}^{\infty} n\left|\sum_{v=1}^{k}(b_{nv} - b_{n+1,v})\right| = \sup_{k\geq 2}[|a|] = |a| < \infty.$$

So, the operator $B(a)$ is bounded.

This completes the proof.

*** Theorem 2.2**

$\sigma(B(a), h) = \{\lambda \in \mathbb{C}: |\lambda - a| \leq 0\}.$

Proof. The proof is similar to that in [9].

*** Theorem 2.3**

The operator $B(a)$ has eigenvalues in h, and $\sigma_p(B(a), h) = \{\lambda \in \mathbb{C}: \lambda = a\}$

*** Proof**

Suppose $B(a)x = \lambda x$ for $x = (x = (x_k)_{k=1}^{\infty} \in h, x \neq \theta$. Then by solving the system of equations

$$\begin{aligned} ax_1 &= \lambda x_1 \\ ax_2 &= \lambda x_2 \\ ax_3 &= \lambda x_3 \\ &\vdots \\ ax_{k+1} &= \lambda x_{k+1} \\ &\vdots \end{aligned}$$

we obtain

$(a-\lambda)x_k = 0, k \geq 1$.
If $\lambda = a$, we have $x \neq \theta$ for all , $k \geq 1$. But, if $\lambda \neq a$, we would have $x = \theta$ for all $k \geq 1$. It follows that $\lambda = a$ is an eigenvalue of $B(a)$, therefore $\sigma_p(B(a), h) = \{\lambda \in \mathbb{C}: \lambda = a\}$.

By Theorems 1.9 and 2.3, we can conclude this result.

*** Corollary 2.1**

$\ker(B(a) - aI)$ reduces $B(a)$.

*** Proof**

Since $\sigma_p(B(a), h) = \{\lambda \in \mathbb{C}: \lambda = a\}$, then $\ker(B(a) - aI) \neq \{0\}$, which leads to
$\ker(B(a) - aI)$ reduces $B(a)$.
Also, we have the following results.

*** Corollary 2.2**

$\sigma_p(B(a), h) = \partial(\sigma(B(a), h))$.

*** Corollary 2.3**

If $\lambda \notin \sigma_p(B(a), h)$, then λ is not isolated point in $\sigma(B(a), h)$.

*** Theorem 2.4**

$\sigma_r(B(a), h) = \emptyset$.

*** Proof**

The proof is trivial by Theorem 1.5.

As a result of the above, since $\sigma(B(a), h) \neq \sigma_r(B(a), h)$, this shows that the converse of Theorem 1.7 is not necessarily true.
The following lemma is stated here in order to have a convenient reference for the proof to come.

*** Lemma 2.1**

If T is a bounded linear operator on a Banach space X into itself, then
$\sigma_r(T, X) = \sigma_p(T^*, X^*) \backslash \sigma_p(T, X)$.

*** Theorem 2.5**

The point spectrum of the adjoint operator $B(a)^*$ on h^* is given by $\sigma_p(B(a)^*, h^*) = \{\lambda \in \mathbb{C}: \lambda = a\}$.

*** Proof**

By Lemma 2.1, we have
$\sigma_p(B(a)^*, h^*) = \sigma_p(B(a), h) \cup \sigma_r(B(a), h)$
$= \{\lambda \in \mathbb{C}: \lambda = a\} = \{a\}$.

Also, we can get the proof if we suppose that $B(a)^* f = \lambda f$ for $f = (f_1, f_2, f_3, \ldots) \neq \theta$ in $h^* \cong \sigma_\infty$ and then solve the system of equations

$$\begin{aligned} (r-\lambda)f_1 &= 0f_2 \\ (r-\lambda)f_2 &= 0f_3 \\ &\vdots \\ (r-\lambda)f_k &= 0f_{k+1} \end{aligned}$$

Then we continue with the same steps in the proof of Theorem 2.3.
Now, we derive the result concerning the continuous spectrum B(a) on h.

Theorem 2.6. The continuous spectrum of the operator $B(a)$ on h is

$$\sigma_c(B(a), h) = \{\lambda \in \mathbb{C}: |\lambda - a| \leq 0\} \backslash \{a\} = \{\lambda \in \mathbb{C}: |\lambda - a| < 0\}.$$

Proof. Since $\sigma(B(a), h)$ is the union of the disjoint sets $\sigma_p(B(a), h)$, $\sigma_r(B(a), h)$ and $\sigma_c(B(a), h)$, then Theorems 2.2, 2.3 and 2.4 imply $\sigma_c(B(a), h) = \{\lambda \in \mathbb{C}: |\lambda - a| < 0\}$.

Corollary 2.4. If $\lambda \neq a$ and $|\lambda - a| < 0\}$ for all $\lambda \in \mathbb{C}$, then $\overline{(B(a) - \lambda I)h} = h$.

Proof We can immediately get the proof from Theorems 1.4, 2.3 and 2.6.

Indeed, the following results can be obtained by the preceding relations :

$I_3(B(a), h) \cup II_3(B(a), h) \cup III_3(B(a), h) = \{a\}$,

Also

$II_2(B(a), h) = \{\lambda \in \mathbb{C}: |\lambda - a| < 0\}$,

Moreover

$III_1(B(a), h) = III_2(B(a), h) = \emptyset$.

*** Theorem 2.7**

The following statements hold:

(i) $\sigma_{ap}(B(a), h) = \sigma(B(a), h)\backslash III_1(B(a), h) = \{\lambda \in \mathbb{C}: |\lambda - a| \leq 0\}$,

(ii) $\sigma_{co}(B(a), h) = \sigma_p(B(a)^*, h^*) = \{a\}$,

(iii) $\sigma_\delta(B(a), h) = \sigma(B(a), h)\backslash I_3(B(a), h)$.

The proof is easy.

*** Recommendation**

After all these new results related to the operator B(a), that we obtained in this work and which led us to new results regarding the subnormal operator in general, we propose to devise an operator, which is represented by certain infinite lower or upper triangular double-band matrix as an example of the subnormal operator and study its fine spectrum on some spaces in several ways to reach more accurate results that can be applied in many situations.

*** Conclusion**

This work has aimed to clarify some general properties of the subnormal operator and derive corresponding results of a generalized difference operator as an example of this type of the operators specifically.

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